Applied mathematics in Engineering, Management and Technology 3(4) 2015:212-220
www.amiemt-journal.com

Positivity preserving schemes with application to finance: option pricing

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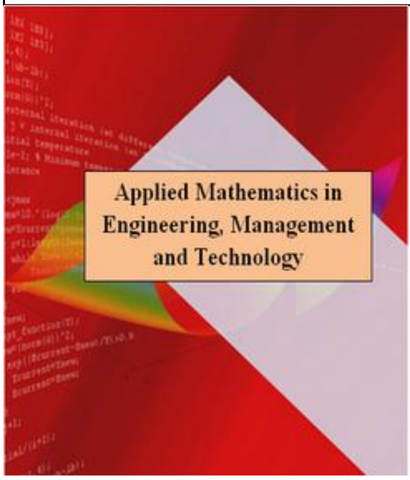

Abstract

Classical methods based on finite difference equation guarantee linear stability, consistency and convergency of the discrete solution to the exact one. But usually the essential qualitative properties of the solution are not transferred to the numerical solution. Thus, the stated disadvantage might be calastrophoric. For example when one solves the Black-Scholes partial differential equation (PDE) it is of great important that numerical scheme be free of spurious oscillations and satisfy the positivity requirement. In this paper, first we show that the positivity is not ensured whit a class of standard finite difference method (we refer to it θ-method) when applied to the Black-Scholes equation for very small time steps. Next, by reforming the discretization of the reaction term of equation a class of method is derived that is free of spurious oscillations around discontinuities and preserving positivity.

Keywords: Finite differences, M-matrix, Black-Scholes equation, θ-method, Positivity preserving, Nonstandard discretization.

Introduction

Mathematical finance is a field of applied mathematics, concerned with financial markets. In the market of financial derivatives the most important problem is the so called *option valuation problem*, i.e. to compute a fair value for the option. The solution of the Black-Scholes partial differential equation determines the option price, respectively according to the used initial conditions. In the computation of the fair price of an option, it is a natural demand that the resulting numerical approximations, should be non-negative. Numerical methods based on standard finite difference approach e.g. fully implicit, Crank-Nicolson and semi implicit schemes (substituting $\theta = 1, 1/2, 0$ respectively in θ-method) are powerful tool for pricing, for more details see [15]. They are usually consistent with the original differential equation and guarantee convergency of the discrete solution to the exact one, but in the presence of discontinuous payoff and low volatility, essential qualitative properties of the solution are not transferred to the numerical solution. Spurious oscillations and negative values might be occurred in the solution [10,11,12,13,15]. Therefore, we need to construct positivity preserving schemes [4, 10, 11, 13, 15], that avoid unrealistic negative values for the solution. One possibility is to use the tool of nonstandard discretization [3, 5, 6, 7, 9, 12, 14].
Consists of the renormalization of denominator of the discrete derivative and the nonlocal approximation of the reaction term in the Black-Scholes equation.
This work is organized as follows: first we introduce in Section 2 the Black-Scholes equation for pricing European option and the θ-method. We review the numerical results of the θ-method for different values of θ in the presence of discontinuities in the initial conditions of the Black-Scholes equation. Next, in Section 3 we propose a modification of the θ-method that enables us to solve accurately the Black-Scholes equation. Furthermore, we investigate the positivity and the stability requirements. Finally, we end the paper with some conclusions in Section 4.

Option pricing

The famous Black-Scholes equation [2, 8] is an effective model for option pricing. The value of a call option is denoted by V and depends on the current market price of the underlying asset, S and the calendar time t : $V = V(S, t)$. The European option pricing in the form of initial value problem can be written as:

$$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad (1)$$

$$V(S,0) = \max(S-K,0)1_{[L,U]}(S),$$
$$V(S,t) \to 0 \text{ as } S \to 0 \text{ or } S \to \infty, \quad (2)$$

with updating of the initial condition at the monitoring dates t_i , $i = 1, \cdots, F$:

$$V(S,t_i) = V(S,t_i^-)1_{[L,U]}(S), \qquad 0 = t_0 < t_1 < \cdots < t_F = T, \quad (3)$$

where $1_{[L,U]}(S)$ is the indicator function, i.e.,

$$1_{[L,U]}(S) = \begin{cases} 1 & if \quad S \in [L,U] \\ 0 & if \quad S \notin [L,U] \end{cases}, \quad (4)$$

here, the parameter $r > 0$ is the interest rate and the reference volatility is $\sigma > 0$.
The θ-method is defined by replacing derivatives with respect to S by

$$\frac{\partial V}{\partial S} \approx \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S},$$

$$\frac{\partial^2 V}{\partial S^2} \approx (1-\theta)\frac{V_{j+1}^{n} - 2V_j^{n} + V_{j-1}^{n}}{\Delta S^2} + \theta\frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta S^2},$$

and the derivative with respect to t by

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^{n}}{\Delta t}.$$

Therefore, the family of standard θ-method for solving (1) lead to a difference equation

$$AV^{n+1} = BV^{n}, \quad (5)$$

where A and B are the following tridiagonal matrices

$$A = tridiag\left\{\frac{rS_j}{2\Delta S} - \frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2, \frac{1}{\Delta t} + \theta\left(\frac{\sigma S_j}{\Delta S}\right)^2 + r, -\frac{rS_j}{2\Delta S} - \frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\right\},$$

$$B = tridiag\left\{\frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2, \frac{1}{\Delta t} - (1-\theta)\left(\frac{\sigma S_j}{\Delta S}\right)^2, \frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\right\}.$$

As we can see in Figures 1-3, the θ-method (for different values of θ) gives negative values and spurious oscillations whenever the parameters σ and r satisfy the relationship $\sigma^2 \ll r$. The parameters used in these simulations,

$$L = 90,\ K = 100,\ U = 110,\ r = 0.05,\ \sigma = 0.001,\ T = 1,\ S_{\max} = 200,$$

have been taken from [13].

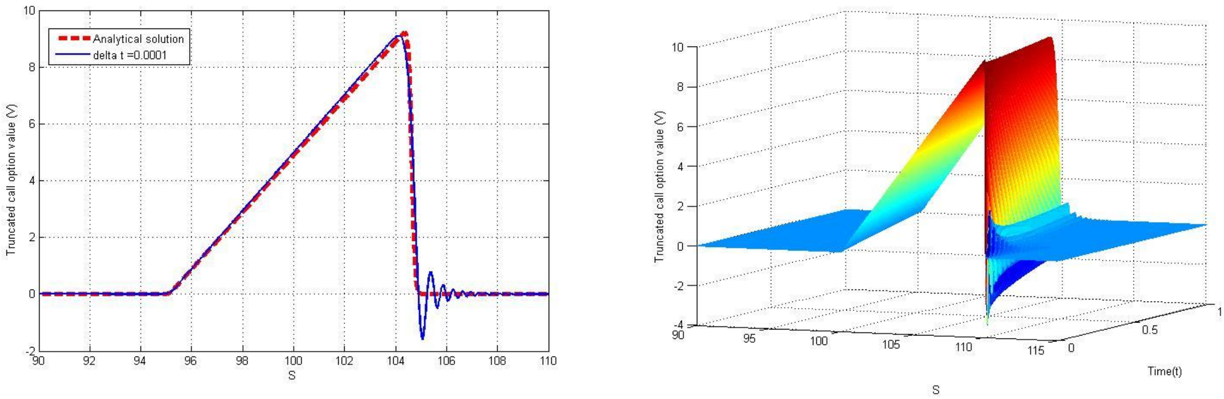

Figure 1. Truncated call option value for the θ-method with $\Delta S = 0.05$, $\Delta t = 10^{-4}$ and $\theta = 0$.

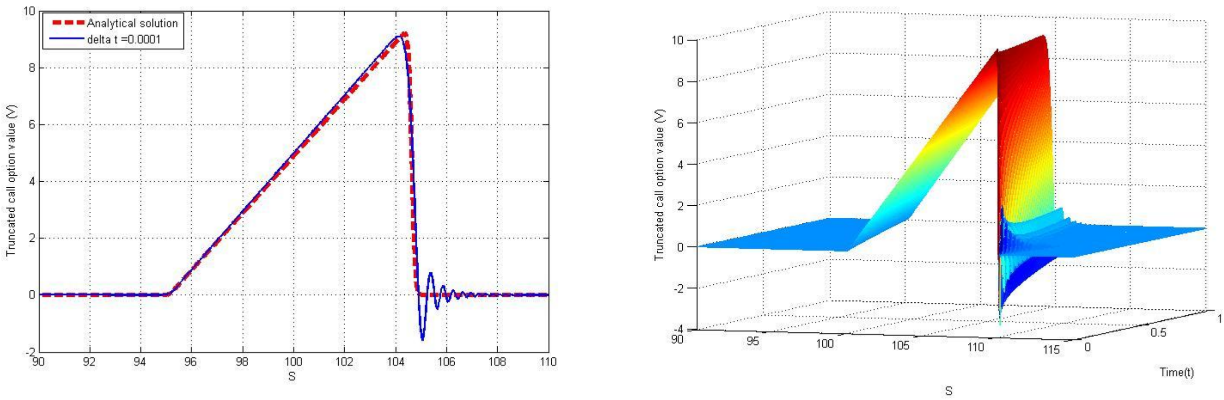

Figure 2. Truncated call option value for the θ-method with $\Delta S = 0.05$, $\Delta t = 10^{-4}$ and $\theta = \frac{1}{2}$.

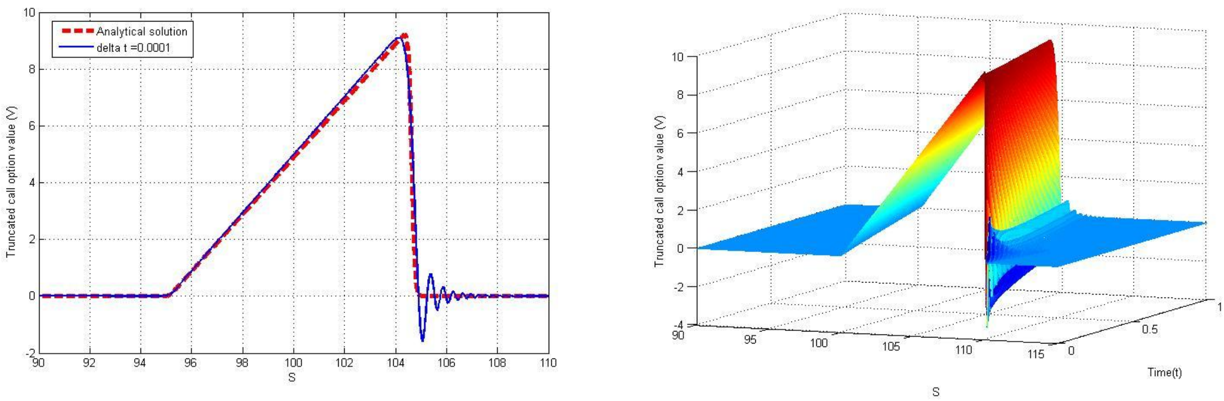

Figure 3. Truncated call option value for the θ-method with $\Delta S = 0.05$, $\Delta t = 10^{-4}$ and $\theta = 1$.

Scheme construction

In order to derive the new accurate scheme which is positivity preserving, we replace the reaction term in (1) by

$$a\left(V_{j+1}^{n+1}+V_{j-1}^{n+1}\right)+b\left(V_{j+1}^{n}+V_{j-1}^{n}\right)+\left(1-2a-2b\right)V_{j}^{n}. \tag{6}$$

Here, a and b are arbitrary constants to be determined below and $\frac{\partial V}{\partial S}$ is discretized through a backward difference. The finite difference approximation provides the equation difference

$$PV^{n+1}=NV^{n}, \tag{7}$$

with P and N are the following tridiagonal matrices:

$$P=tridiag\left\{ra+\frac{rS_j}{\Delta S}-\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2,\frac{1}{\Delta t}-\frac{rS_j}{\Delta S}+\theta\left(\frac{\sigma S_j}{\Delta S}\right)^2,ra-\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\right\}, \tag{8}$$

$$N=tridiag\left\{\frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2-rb,\frac{1}{\Delta t}-(1-\theta)\left(\frac{\sigma S_j}{\Delta S}\right)^2-r\left(1-2a-2b\right),\frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2-rb\right\}. \tag{9}$$

Theorem 1. Sufficient conditions for scheme (7) to be positive are,

$$a\le-\frac{r}{2\sigma^2\theta}\quad,\quad b\le\frac{1-\theta}{2r}\sigma^2\quad,\quad \Delta t<\frac{1}{(1-\theta)(\sigma M)^2+r\left(1-2a-2b\right)}. \tag{10}$$

Proof. From (7) it is enough to show that $P^{-1}\ge 0$ and $N\ge 0$.
If P be an M-matrix [16], then $P^{-1}\ge 0$ [1], therefore we have to put

$$ra+\frac{rS_j}{\Delta S}-\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\le 0 \quad\Rightarrow\quad ra\le\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2-\frac{rS_j}{\Delta S}, \tag{11}$$

$$ra-\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2\le 0 \quad\Rightarrow\quad ra\le\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2. \tag{12}$$

From (11) and (12) we can write

$$ra\le\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2-\frac{rS_j}{\Delta S},$$

$$\Leftrightarrow a\le\frac{1}{r}\left[\frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2-\frac{rS_j}{\Delta S}\right],$$

$$\Leftrightarrow a \le \frac{\theta\sigma^2}{2r}\left[\left(\frac{S_j}{\Delta S}\right)^2 - \frac{r}{\sigma^2}\left(\frac{S_j}{\Delta S}\right) + \frac{r^2}{\sigma^4\theta^2} - \frac{r^2}{\sigma^4\theta^2}\right], \tag{13}$$

$$\Leftrightarrow a \le \frac{\theta\sigma^2}{2r}\left[\left(\frac{S_j}{\Delta S} - \frac{r}{\theta\sigma^2}\right)^2 - \frac{r^2}{\sigma^4\theta^2}\right],$$

$$\Leftrightarrow a \le \frac{\theta\sigma^2}{2r}\left(\frac{S_j}{\Delta S} - \frac{r}{\theta\sigma^2}\right)^2 - \frac{r}{2\sigma^2\theta},$$

now, the last inequality in (13) shows sufficiency of $a \le -\frac{r}{2\sigma^2\theta}$ for (10).

In order to nonnegativity for N , we write

$$\frac{1-\theta}{2}\left(\frac{\sigma S_j}{\Delta s}\right)^2 - rb \ge 0 \quad \Rightarrow \quad b \le \frac{1-\theta}{2r}\sigma^2,$$

also we have

$$\frac{1}{\Delta t} - (1-\theta)\left(\frac{\sigma S_j}{\Delta S}\right)^2 - r(1-2a-2b) \ge 0,$$

then

$$\Delta t < \frac{1}{(1-\theta)(\sigma M)^2 + r(1-2a-2b)}.$$

This completes the proof. □

Theorem 2. The new scheme is conditionally stable and convergent with local truncation error $O(\Delta t, \Delta S^2)$.

Proof. Under conditions (10) $P = [p_{ij}]$ is similar to a symmetric tridiagonal matrix (see [16], p. 24), so that the eigenvalues of P ($\lambda_i(P); i = 1, \cdots, N$) are real. Also P is row diagonally dominant with

$$\delta_i = |p_{ii}| - \sum_{j \ne i} |p_{ij}| = \frac{1}{\Delta t} + 2ra > 0 .$$

So $\left\|P^{-1}\right\|_\infty \le \dfrac{1}{\frac{1}{\Delta t}+2ra}$ (see [16], p. 8) and by combining with $\left\|N\right\|_\infty = \frac{1}{\Delta t}+2ra-r$, we have

$$\rho\left(P^{-1}N\right) \le \left\|P^{-1}N\right\|_\infty \le \left\|P^{-1}\right\|_\infty \left\|N\right\|_\infty \le \frac{\frac{1}{\Delta t}+2ra-r}{\frac{1}{\Delta t}+2ra} < 1,$$

where $\rho\left(P^{-1}N\right)$ is the spectral radius of the matrix $P^{-1}N$. Therefor the scheme is stable and then via the Lax-theorem convergent with local truncation error:

$$T_j^n = -\frac{V(S_j,t_{n+1})-V(S_j,t_n)}{\Delta t} + rS_j\frac{V(S_j,t_{n+1})-V(S_{j-1},t_{n+1})}{\Delta S}$$

$$+\frac{1-\theta}{2}(\sigma S_j)^2\left(\frac{V(S_{j-1},t_n)-2V(S_j,t_n)+V(S_{j+1},t_n)}{\Delta S^2}\right)$$

$$+\frac{\theta}{2}(\sigma S_j)^2\left(\frac{V(S_{j-1},t_{n+1})-2V(S_j,t_{n+1})+V(S_{j+1},t_{n+1})}{\Delta S^2}\right)$$

$$-r\left(a\left(V(S_{j+1},t_{n+1})+V(S_{j-1},t_{n+1})\right)+b\left(V(S_{j+1},t_n)+V(S_{j-1},t_n)\right)+(1-2a-2b)V(S_j,t_n)\right), \quad (14)$$

by Taylor's expansion

$$V(S_j,t_{n+1}) = V(S_j,t_n)+\Delta t\left(\frac{\partial V}{\partial t}\right)_j^n+\frac{1}{2}\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n+\frac{1}{6}\Delta t^3\left(\frac{\partial^3 V}{\partial t^3}\right)_j^n+\cdots,$$

$$V(S_{j+1},t_n) = V(S_j,t_n)+\Delta S\left(\frac{\partial V}{\partial S}\right)_j^n+\frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n+\frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n+\cdots,$$

$$V(S_{j+1},t_n)=V(S_j,t_n)-\Delta S\left(\frac{\partial V}{\partial S}\right)_j^n+\frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n-\frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n+\cdots,$$

$$V(S_{j+1},t_{n+1})=V(S_j,t_n)+\Delta S\left(\frac{\partial V}{\partial S}\right)_j^n+\Delta t\left(\frac{\partial V}{\partial t}\right)_j^n+\frac{\Delta S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n+\frac{\Delta t^2}{2}\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n+\Delta S\,\Delta t\left(\frac{\partial V}{\partial S\,\partial t}\right)_j^n+\cdots,$$

$$V(S_{j+1},t_{n+1})=V(S_j,t_n)-\Delta S\left(\frac{\partial V}{\partial S}\right)_j^n+\Delta t\left(\frac{\partial V}{\partial t}\right)_j^n+\frac{\Delta S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n+\frac{\Delta t^2}{2}\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n-\Delta S\,\Delta t\left(\frac{\partial V}{\partial S\,\partial t}\right)_j^n+\cdots,$$

by substitution into (14) we have

$$\begin{aligned}T_j^n=&\left(-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-rV\right)_j^n-\frac{1}{2}\left(\frac{1}{2}+2ra\right)\Delta t\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n+rj\,\Delta S\,\Delta t\left(\frac{\partial V}{\partial S\,\partial t}\right)_j^n\\&-ra\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n+\left(\frac{1}{2}j+ra+rb\right)\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n+\cdots.\end{aligned} \tag{15}$$

but V is the solution of the Black-Scholes equation, therefore, we have

$$\left(-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-rV\right)_j^n=0.$$

hence the new scheme is consistent with the (1) and $T_j^n=O\left(\Delta t,\Delta S^2\right)$. This completes the proof. □

In Figures 4-6 numerical results for the proposed modification of θ-method are shown with $\sigma^2 \ll r$ and different values of θ. Comparing with Figures 1-3 we observe that the new schemes perform very well. The parameters used here are

$$L=90,\ K=100,\ U=110,\ r=0.05,\ \sigma=0.001,\ T=0.01,\ S_{\max}=120.$$

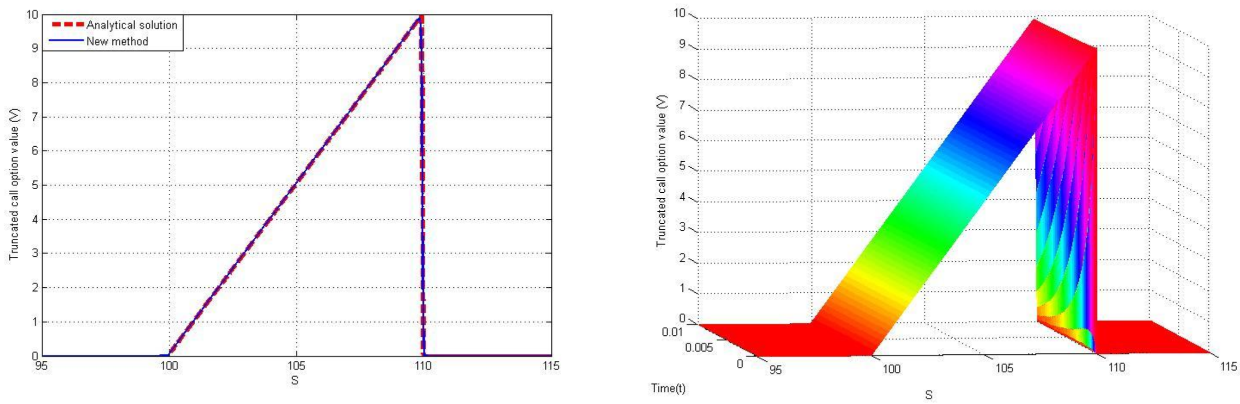

Figure 4. Truncated call option value for the new scheme with $\Delta S=0.01$, $\Delta t=10^{-6}$ and $\theta=0$.

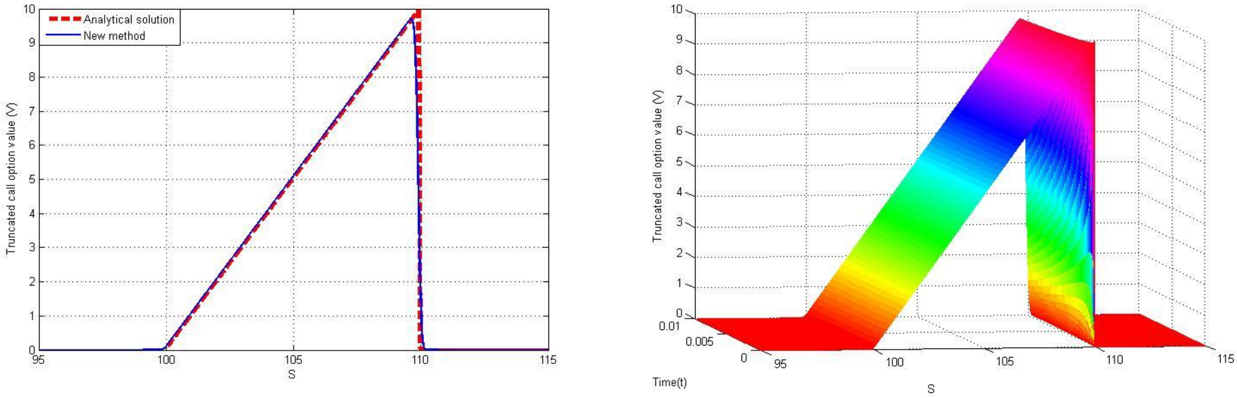

Figure 5. Truncated call option value for the new scheme with $\Delta S = 0.01$, $\Delta t = 10^{-6}$ and $\theta = \frac{1}{2}$.

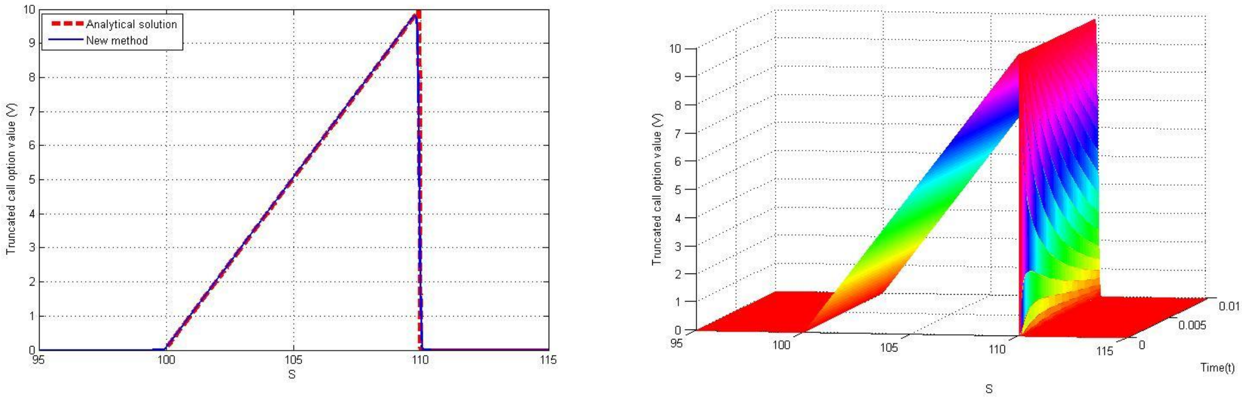

Figure 6. Truncated call option value for the new scheme with $\Delta S = 0.01$, $\Delta t = 10^{-6}$ and $\theta = 1$.

Conclusion

We constructed a modification of θ-method based on a nonstandard discretization of the reaction term in Black-Scholes equation. The proposed nonstandard scheme is free of spurious oscillations and satisfies the positivity requirement as is demanded for the financial solution of the Black-Scholes equation. Future work will include extending the new nonstandard discretization scheme to nonlinear Black-Scholes equation.

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